

On semilinear elliptic equations with global coupling

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Abstract

We consider the problem

$$\begin{cases} \Delta A - A + A^p - kA \int_{\mathbb{R}^n} A^2 dx = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} A(x) = 0, \end{cases}$$

where $p > 1$, $k > 0$ are constants. We classify the existence of all possible positive solutions to this problem.

1 Introduction

Equations with global coupling arise in the study of pattern formation in various fields of science such as fluid mechanics as well as chemistry or biology. Especially Riecke [12] has suggested the following equation:

$$A_t = \Delta A + \mu A + c|A|^2 A - |A|^4 A - k \int_{\mathbb{R}^n} |A|^2 dx, \quad (1)$$

where A is a complex-valued function defined on $\mathbb{R}^n \times (0, \infty)$, $k > 0$ and c, μ are real numbers.

We are interested in positive solutions. This means that we replace (1) by the following equation:

$$A_t = \Delta A + \mu A + cA^3 - A^5 - kA \int_{\mathbb{R}^n} A^2 dx, \quad (2)$$

where A is a positive function defined on $\mathbb{R}^n \times (0, \infty)$.

Wei and Winter [13] studied this equation (2), together with the steady-state solutions of (2) which satisfy

$$\Delta A + \mu A + cA^3 - A^5 - kA \int_{\mathbb{R}^n} A^2 dx = 0. \quad (3)$$

They have classified the existence of positive solutions to (3) and studied the stability of all possible standing wave solutions. For other researches for equations concerned with pattern formation, consult Matthews and Cox [7] and Norbury, Wei and Winter [9].

In this present paper, we shall consider the following problem

$$\begin{cases} \Delta A - A + A^p - kA \int_{\mathbb{R}^n} A^2 dx = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} A(x) = 0, \quad A(0) = \max_{x \in \mathbb{R}^n} A(x), \end{cases} \quad (4)$$

where A is a positive-valued function defined on $\mathbb{R}^n \times (0, \infty)$ and $p > 1, k > 0$ are constants, n is the space dimension $n \geq 1$, integer.

This equation is from (3), modified and generalized to seize mathematical structures. We shall find classical positive solutions to (4), using the uniqueness result to single power nonlinear equation given by Kwong [5].

Following is the main result of the present paper:

Theorem 1. *For any $p \in (0, p^*(n))$, the problem (4) has a positive radially symmetric solution if $k > 0$ is sufficiently small.*

For any $p \in [p^(n), \infty)$, the problem (4) does not have any solutions for any $k > 0$.*

Here, $p^(n)$ is the exponent called Sobolev exponent defined by the following formula:*

$$p^*(n) = \begin{cases} \infty & n = 1, 2, \\ \frac{n+2}{n-2} & n \geq 3. \end{cases}$$

In the next section we present the proof of the theorem and more detailed formulation of the theorem. We note here that the method is essentially based on Wei and Winter [13].

2 Proof of the theorem

First we state the key lemma for the proof.

Lemma 1. *Consider the problem*

$$\begin{cases} \Delta A - \omega A + A^p = 0, \\ \lim_{|x| \rightarrow \infty} A(x) = 0, \quad A(0) = \max_{x \in \mathbb{R}^n} A(x), \end{cases} \quad (5)$$

where $p > 1$ and $\omega > 0$.

1. If $1 < p < p^*(n)$, the problem (5) possesses the unique positive radially symmetric classical solution for each $\omega > 0$.

Moreover, the solution A is given by the following formula:

$$A(x) = \omega^{\frac{1}{p-1}} A_0(\sqrt{\omega}x), \quad (6)$$

where A_0 is the unique solution of the following problem:

$$\begin{cases} \Delta A - A + A^p = 0, \\ \lim_{|x| \rightarrow \infty} A(x) = 0, \quad A(0) = \max_{x \in \mathbb{R}^n} A(x), \end{cases} \quad (7)$$

(This problem (7) is the problem (5) with $\omega = 1$.)

- 2. If $p^*(n) \leq p < \infty$ the problem (5) does not have any positive solutions.
- 3. The function A_0 has exponential decay at infinity:

$$|A_0(x)| \leq C e^{-\delta|x|}, \quad x \in \mathbb{R}^n,$$

for some $C, \delta > 0$.

Proof. The existence and non-existence part is due to the classical work of Pohozev [11]. The fact that for $p \geq p^*(n)$, (5) does not have any solution follows from a well known identity of Pohozaev. Also see Berestycki and Lions [1] and Berestycki, Lions and Peletier [2].

The decay rate of A_0 is studied in Berestycki and Lions [1].

The classical work of Gidas, Ni and Nirenberg [3, 4] tells us that all positive solutions are radially symmetric.

The uniqueness result of (7) is by Kwong [5] for $n \geq 3$, by Mcleod and Serrin [8] for $n = 2$, and by Berestycki and Lions [1] for $n = 1$. It is enough to prove uniqueness for (5) with any $\omega > 0$, and the representation formula (6).

$A(x)$ defined by the formula (6) is a solution to the problem (5). On the other hand, for the solution $B(x)$ to (5), $B_0(x) := \omega^{-\frac{1}{p-1}} B(\frac{x}{\sqrt{\omega}})$ is a solution to (7). By the uniqueness theorem for (7), this B_0 is equal to A_0 at any point. This fact completes the proof. \square

First we note that the problem (4) is equivalent to

$$\begin{cases} \Delta A - \omega A + A^p = 0, \\ \omega = 1 + k \int_{\mathbb{R}^n} A^2 dx, \\ \lim_{|x| \rightarrow \infty} A(x) = 0, \quad A(0) = \max_{x \in \mathbb{R}^n} A(x). \end{cases}$$

So the solution of (4) is of the form

$$A(x) = \omega^{\frac{1}{p-1}} A_0(\sqrt{\omega}x),$$

with the consistency condition

$$\begin{aligned} \omega &= 1 + k \int_{\mathbb{R}^n} \omega^{\frac{2}{p-1}} A_0(\sqrt{\omega}x)^2 dx \\ &= 1 + k \int_{\mathbb{R}^n} \omega^{\frac{2}{p-1} - \frac{n}{2}} A_0(y)^2 dy. \end{aligned} \quad (8)$$

The relation (8) is equivalent to

$$k\alpha = (\omega - 1)\omega^{\frac{n}{2} - \frac{2}{p-1}},$$

where

$$\alpha = \int_{\mathbb{R}^n} A_0(y)^2 dy.$$

Note that $0 < \alpha < \infty$ from the fact 3. of the Lemma 1.

We analyze the function

$$f(\omega) := (\omega - 1)\omega^{\frac{n}{2} - \frac{2}{p-1}}.$$

We define $e_{n,p} := \frac{n}{2} - \frac{2}{p-1}$. Five cases occur when we investigate the function $f(\omega)$ in $(0, \infty)$:

$$1 \quad e_{n,p} + 1 < 0 \quad \text{i.e. } 1 < p < 1 + \frac{4}{n+2}.$$

$f(\omega)$ attains its maximum at $\omega_{n,p} := \frac{-e_{n,p}}{-e_{n,p} - 1} > 0$, and the maximum is positive.

$$f(+0) := \lim_{\omega \rightarrow 0} f(\omega) = -\infty, \quad f(\infty) := \lim_{\omega \rightarrow \infty} f(\omega) = 0.$$

$$2 \quad e_{n,p} + 1 = 0 \quad \text{i.e. } p = 1 + \frac{4}{n+2}.$$

$f(\omega)$ is increasing in $(0, \infty)$.
 $f(+0) = -\infty, \quad f(\infty) = 1$.

$$3 \quad 0 < e_{n,p} + 1 < 1 \quad \text{i.e. } 1 + \frac{4}{n+2} < p < 1 + \frac{4}{n}.$$

$f(\omega)$ is increasing in $(0, \infty)$.
 $f(+0) = -\infty, \quad f(\infty) = \infty$.

$$4 \quad e_{n,p} + 1 = 1 \quad \text{i.e. } p = 1 + \frac{4}{n}.$$

$f(\omega)$ is increasing in $(0, \infty)$.
 $f(+0) = -1, \quad f(\infty) = \infty$.

$$5 \quad 1 < e_{n,p} + 1 \quad \text{i.e. } 1 + \frac{4}{n} < p.$$

$f(\omega)$ attains its minimum at $\omega_{n,p} = \frac{e_{n,p}}{e_{n,p} + 1} > 0$, and the minimum is negative.

$$f(+0) = 0, \quad f(\infty) = \infty.$$

Following is the detailed statement of the theorem:

Theorem 2. Let the space dimension $n \geq 1$ be fixed.

$$1 \quad 1 < p < 1 + \frac{4}{n+2}.$$

1-1 If $k > \frac{f(\omega_{n,p})}{\alpha}$, then the problem (4) does not have any solution.

1-2 If $k = \frac{f(\omega_{n,p})}{\alpha}$, then the problem (4) has exactly one solution.

1-3 If $\frac{f(\omega_{n,p})}{\alpha} > k > 0$, then the problem (4) has two solutions.

$$2 \quad p = 1 + \frac{4}{n+2}.$$

2-1 If $k \geq 1$, then the problem (4) does not have any solution.

2-2 If $0 < k < 1$, then the problem (4) has exactly one solution.

$$3 \quad 1 + \frac{4}{n+2} < p < p^*(n).$$

3-1 For any $k > 0$, the problem (4) has exactly one solution.

$$4 \quad p^*(n) \leq p.$$

4-1 For any $k > 0$, the problem (4) does not have any solution.

Here a solution means a positive radial symmetric classical solution.

Proof of Theorem 2. For $p \geq p^*(n)$. Suppose there is a solution $A(x)$, for contradiction. Then A is a solution of

$$\Delta A - \omega A + A^p = 0 \quad (9)$$

with

$$\omega := 1 + k \int_{\mathbb{R}^n} A^2 dx. \quad (10)$$

This contradicts to the fact 2. of the Lemma 1.

From now on we concentrate on the case $1 < p < p^*(n)$.

Lemma 2. Suppose that for $k > 0$, there exists $\omega > 0$ such that $f(\omega) = k\alpha$. Then $A(x) := \omega^{\frac{1}{p-1}} A_0(\sqrt{\omega}x)$ is a solution to (4).

On the other hand, for a solution $A(x)$ of (4) with $k > 0$, there exists $\omega > 0$ such that $f(\omega) = k\alpha$. Moreover, A is formulated with this ω : $A(x) = \omega^{\frac{1}{p-1}} A_0(\sqrt{\omega}x)$.

Proof. First remember that for $A(x) := \omega^{\frac{1}{p-1}} A_0(\sqrt{\omega}x)$, $f(\omega) = k\alpha$ is equivalent to $1 + \int_{\mathbb{R}^n} A^2 dx = \omega$.

Now we prove the first statement. The left hand side of the equation in problem (4) is calculated in the following way:

$$\begin{aligned}\Delta A - A + A^p - kA \int_{\mathbb{R}^n} A^2 dx &= \Delta A - \left(1 + \int_{\mathbb{R}^n} A^2 dx\right) A + A^p \\ &= \Delta A - \omega A + A^p \\ &= 0.\end{aligned}$$

Then we prove the last statement. A solution A of (4) is a solution of (9) with (10). Here ω has to be positive, for otherwise the equation (9) cannot have any solution. From (9) comes the formula $A(x) = \omega^{\frac{1}{p-1}} A_0(\sqrt{\omega}x)$, and the relation $f(\omega) = k\alpha$ is from (10). \square

The above lemma and the observation of the graph of $f(u)$ completes the proof of Theorem 2. \square

Remark 1. Note that any solution of our problem (4) is in the form (6): these solutions converges to A_0 pointwise as k in (4) goes to zero.

Remark 2. We can treat a more generalized problem in the same way:

$$\begin{cases} \Delta A - A + A^p - kA \int_{\mathbb{R}^n} A^r dx = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} A(x) = 0, \quad A(0) = \max_{x \in \mathbb{R}^n} A(x), \end{cases}$$

where r is positive. We need three cases to be dealt with if $n \geq 3$:

$$0 < r < \frac{2n}{n-2}, \quad \frac{2n}{n-2} \leq r < \frac{2(n+2)}{n-2}, \quad \frac{2(n+2)}{n-2} \leq r.$$

Our $r = 2$ is always in the first case, which is most bothersome. We shall omit the full observation.

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